DIFFRACTION FROM A HALF-SURFACE OF WAVES FORMED ON THE SURFACE OF A LIQUID BY A PERIODICALLY ACTING SOURCE

(DIFRAKTSIIA OT POLUPLOSKOSTI VOLN, OBRAZUEMYKH NA POVERKHNOSTI ZHIDKOSTI PERIODICHESKI DEISTVUIUSHCHIM ISTOCHNIKOM)

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1. Statement of the problem and derivation of its solution. An infinitely deep, ideal, heavy liquid fills the half-space $z < 0$. In the liquid is submerged a plane, semi-infinite, vertical wall, the edge of which coincides with the axis Oz. Under the surface of the liquid at several points characterized by the cylindrical coordinates $(r', a, -h)$ are located sources acting periodically with frequency σ and having maximum power Q.

We study the diffraction of waves incident on the surface of the liquid. Considering the motion of the liquid to be potential, and bearing in mind its periodic character, we introduce the velocity potential $\phi(r, \theta, z)e^{i\sigma t}$. The function $\phi(r, \theta, z)$ throughout the entire half-space, occupied by the liquid, should satisfy the Laplace equation

$$\Delta\varphi = 0 \qquad (1.1)$$

the condition on the free surface of the liquid

$$\varphi = \frac{g}{\sigma^2}\frac{\partial\varphi}{\partial z} \quad \text{for} \quad z = 0 \qquad (1.2)$$

and the conditions of the solid wall

$$\frac{\partial\varphi}{\partial\theta} = 0 \quad \text{for} \quad \theta = 0 \text{ and } \theta = 2\pi \qquad (1.3)$$

The motion should damp out with increase in depth of the place of observation; therefore

$$\varphi(r, \theta, z) \to 0, \qquad \text{for } z \to -\infty \qquad (1.4)$$

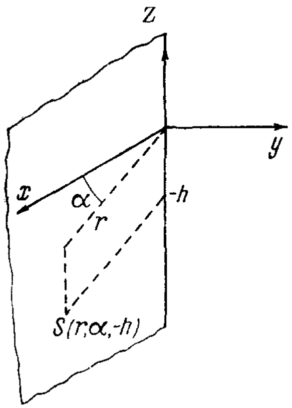

Fig. 1.

Finally, at the point $S(r', \alpha, -h)$ the function $\phi(r, \theta, z)$ should have a singularity of the form

$$\frac{Q}{4\pi}\frac{1}{R_1} \tag{1.5}$$

$$R_1^2 = r^2 + r'^2 - 2rr'\cos(\theta - \alpha) + (z - z')^2$$

The problem consists of finding the solution to the Laplace equation (1.1) which satisfies conditions (1.2) to (1.5). Bearing in mind the integral representation of the potential of a source in an infinite liquid domain

$$\frac{Q}{4\pi}\frac{1}{\sqrt{R^2 + (z+h)^2}} = \frac{Q}{4\pi}\int_0^\infty e^{-k(z+h)} J_0(kR)\,dk \tag{1.6}$$

we shall seek a function $\phi(r, \theta, z)$ in the form

$$\varphi(r, \theta, z) = \frac{Q}{4\pi}\left[\int_0^\infty e^{-k(z+h)}\Psi(k, r, \theta)\,dk + \int_0^\infty e^{kz}A(k)\Psi(k, r, \theta)\,dk\right] \tag{1.7}$$

where the function $A(k)$ is determined in such a way as to satisfy the boundary condition (1.2). Setting $\phi(r, \theta, z)$ in the form (1.7) in Equation (1.2), we obtain the following expression for the function $A(k)$:

$$A(k) = -e^{-kh}\frac{1 + gk/\sigma^2}{1 - gk/\sigma^2} \tag{1.8}$$

Thus, the function $\phi(r, \theta, z)$ for $-z < h$ can be put in the form

$$\varphi(r, \theta, z) = \frac{Q}{4\pi}\left[\int_0^\infty e^{-k(z+h)}\Psi(k, r, \theta)\,dk - \int_0^\infty e^{-k(h-z)}\frac{1 + gk/\sigma^2}{1 - gk/\sigma^2}\Psi(k, r, \theta)\,dk\right. \tag{1.9}$$

The function $\Psi(k, r, \theta)$ is found, following Sommerfeld [1], by constructing the branched solutions of the Laplace equation. It was determined by this method by Sretenskii in [2].

The function $\Psi k k, r, \theta)$ has the following form:

$$\Psi(k, r, \theta) = \begin{cases} J_0(kR) + J_0(k\bar{R}) + V(\theta, r, k) & \text{for} \quad 0 < \theta < \pi - \alpha \\ J_0(kR) + V(\theta, r, k) & \text{for} \quad \pi - \alpha < \theta < \pi + \alpha \\ V(\theta, r, k) & \text{for} \quad \pi + \alpha < \theta < 2\pi \end{cases} \tag{1.10}$$

$$V(\theta,\ r,\ k,) = -\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}[(e^{1/2(\alpha-\theta)i}e^{1/2\eta} + e^{-1/2(\alpha-\theta)i}e^{-1/2\eta})^{-1} + \tag{1.11}$$

$$+ (e^{1/2(\alpha+\theta)i}e^{-1/2\eta} + e^{-1/2(\alpha+\theta)i}e^{1/2\eta})^{-1}]\, J_0(kR\eta)\, d\eta$$

$$R^2 = r^2 + r'^2 - 2rr'\cos(\theta - \alpha)$$
$$\overline{R}^2 = r^2 + r'^2 - 2rr'\cos(\theta + \alpha) \tag{1.12}$$
$$R_\eta^2 = r^2 + r' + 2rr'\cosh\eta$$

To study the form of the disturbed surface of the liquid, we make use of the relation between the displacement of the free surface of the liquid and the velocity potential:

$$\zeta = \frac{i\sigma}{g}\varphi(r,\ \theta,\ 0)\, e^{i\sigma t} \tag{1.13}$$

From this

$$\zeta(r,\ \theta) = \frac{iQ\sigma}{2\pi g}\, e^{i\sigma t}\int\limits_0^\infty \frac{e^{-kh}}{k - \sigma^2/g}\,\Psi(k,\ r,\ \theta)\, k\, dk \tag{1.14}$$

On the basis of Expression (1.10) for $\Psi(k,\ r,\ \theta)$, the displacement of the liquid surface may be put in the form

$$\zeta = \zeta_1 + \zeta_2 + \zeta_3 \quad \text{for} \quad 0 < \theta < \pi - \alpha$$
$$\zeta = \zeta_1 + \zeta_3 \quad \text{for} \quad \pi - \alpha < \theta < \pi + \alpha \tag{1.15}$$
$$\zeta = \zeta_3 \quad \text{for} \quad \pi + \alpha < \theta < 2\pi$$

$$\zeta_1 = \frac{iQ\sigma^3}{2\pi g^2}\, e^{i\sigma t}\int\limits_0^\infty \exp\left(-\frac{\sigma^2 h}{g}\,\xi\right) J_0\left(\frac{\sigma^2}{g}\,R\xi\right)\frac{\xi d\zeta}{\xi - 1} \tag{1.16}$$

$$\zeta_2 = \frac{iQ\sigma^3}{2\pi g^2}\, e^{i\sigma t}\int\limits_0^\infty \exp\left(-\frac{\sigma^2 h}{g}\,\xi\right) J_0\left(\frac{\sigma^2}{g}\,\overline{R}\xi\right)\frac{\xi d\zeta}{\xi - 1} \tag{1.17}$$

$$\zeta_3 = -\frac{iQ\sigma^3}{8\pi^2 g^2}\, e^{i\sigma t}\int\limits_0^\infty \exp\left(-\frac{\sigma^2 h}{g}\,\xi\right)\frac{\xi d\xi}{\xi - 1}\times$$

$$\times\int\limits_{-\infty}^{\infty}\left[\frac{\cos[1/2(\alpha-\theta)]\cosh(1/2\eta)}{\cos^2[1/2(\alpha-\theta)]\cosh^2(1/2\eta) + \sin^2[1/2(\alpha-\theta)]\sinh^2(1/2\eta)} + \right.$$

$$\left. + \frac{\cos[1/2(\alpha+\theta)]\cosh(1/2\eta)}{\cos^2[1/2(\alpha+\theta)]\cosh^2(1/2\eta) + \sin^2[1/2(\alpha+\theta)]\sinh^2(1/2\eta)}\right] J_0\left(\frac{\sigma^2}{g}\,R_\eta\xi\right) d\eta \tag{1.18}$$

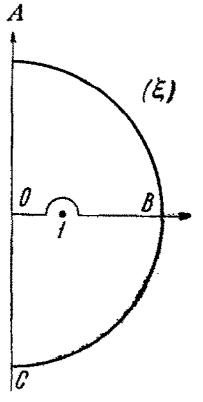

Fig. 2.

The term ζ_1 corresponds to the wave excited at the surface of the liquid, filling the half-space $z < 0$, when the source is at the point $S(r',\ \alpha,\ -h)$. The term ζ_2 corresponds to the wave excited by the source located at the image point $S_1(r',\ -\alpha,\ -h)$. Finally, ζ_3 is the required diffraction.

2. Asymptotic analysis of the solution. We seek the asymptotic formulas for the displacement of the liquid. We start with an analysis of ζ_1. Replacing the Bessel function by the half-sum of the Haenkel functions, we write the integral in Expression (1.16) in the form

$$J=\int_0^\infty \exp\left(-\frac{\sigma^2 h}{g}\,\xi\right) H_0^{(1)}\left(\frac{\sigma^2}{g}R\xi\right)\frac{\xi d\xi}{\xi-1}+\int_0^\infty \exp\left(-\frac{\sigma^2 h}{g}\,\xi\right) H_0^{(2)}\left(\frac{\sigma^2}{g}R\xi\right)\frac{\xi d\xi}{\xi-1} \quad (2.1)$$

Assuming the dimensionless magnitude $\sigma^2 R/g$ to be large, we replace the integration along the real axis by an integration along the path OAB (Fig. 2) in the first term, and along the path OCB in the second term, where AB and CB are arcs of a circle with infinitely large radius.

In the original expression a pole is located along the path of integration at $\xi = 1$.

We assume that the path of integration goes around this pole along a semi-circle of small radius lying in the upper half-plane. As is shown in the analysis below, the radiation condition is satisfied by such a choice of path of integration. In deforming the path of integration in the second term of Formula (2.1) it is necessary to include the contribution near the point $\xi = 1$. Accomplishing the indicated deformation of the path of integration, and taking into account the contribution of the point $\xi = 1$, we transform Expression (2.1) in the form

$$J=-\int_0^\infty \exp\left(-\frac{\sigma^2 h}{g}\,\alpha i\right) H_0^{(1)}\left(i\,\frac{\sigma^2}{g}R\alpha\right)\frac{\alpha d\alpha}{\alpha i-1}-$$
$$-\int_0^{-\infty} \exp\left(-\frac{\sigma^2 h}{g}\,\alpha i\right) H_0^{(2)}\left(i\,\frac{\sigma^2}{g}R\alpha\right)\frac{\alpha d\alpha}{\alpha i-1}-2\pi i\exp\left(-\frac{\sigma^2 h}{g}\right) H_0^{(2)}\left(\frac{\sigma^2}{g}R\right) \quad (2.2)$$

The integrals entering this expression may be asymptotically evaluated by use of the theorem proposed by Sretenskii in [3].* Carrying through the computation, we come to the conclusion that these

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* Let

$$f(\xi)=\sum_{m=1}^{\infty} a_m \xi^{\frac{m}{l}-1} \quad (l>0)$$

Then for large values of ω the asymptotic expression is valid

$$\int^{-\infty} f(\xi)\,H_s^{(2)}(i\xi\omega)\,d\xi=\sum_{m=1}^{\infty}\delta_m a_m \omega^{-\frac{m}{l}} \qquad \left(\delta_m=\int_0^{-\infty}\eta^{\frac{m}{l}-1}H_s^{(2)}(i\eta)\,d\eta\right)$$

integrals are of order $(\sigma^2 R/g)^{-2}$, and therefore

$$J = -2\pi i \exp\left(-\frac{\sigma^2 h}{g}\right) H_0^{(2)}\left(\frac{\sigma^2}{g} R\right) + O(\sigma^2 R/g)^{-2} \tag{2.3}$$

Thus, for large values of the parameter $\sigma^2 R/g$, the part of the displacement due to ζ_1 may be put in the form

$$\zeta_1 = \frac{Q\sigma^3}{2g^2} e^{i\sigma t} \exp\left(-\frac{\sigma^2 h}{g}\right) H_0^{(2)}\left(\frac{\sigma^3}{g} R\right) + (\sigma^2 R/g)^{-2} \tag{2.4}$$

Analogous considerations lead, for large values of the parameter $\sigma^2 R/g$, to the aymptotic value of the quantity

$$\zeta_2 = \frac{Q\sigma^3}{2g^2} e^{i\sigma t} \exp\left(-\frac{\sigma^2 h}{g}\right) H_0^{(2)}\left(\frac{\sigma^2}{g} \overline{R}\right) + O(\sigma^2 \overline{R}/g)^{-2} \tag{2.5}$$

We now pass to the asymptotic approximation of the quantity ζ_3. For this purpose, we first determine the parameter with respect to which the asymptotic approximation is obtained, and consequently we also establish the region in which the asymptotic formula will be valid.

We introduce the dimensionless parameters ρ, ρ', and ρ_1 defined by

$$r = \rho r_0, \quad r' = \rho' r_0, \quad R = \rho_1 r_0$$

We assume that the quantity $\omega = \sigma^2 r_0/g$ is large enough for only the first term in a series in negative powers of ω to be significant in the asymptotic approximation. We also assume that the asymptotic formula will be sought for a position of the point of observation and of the source for which all the parameters ρ, ρ' and ρ_1 may be chosen greater than one.

Reversing the order of integration in Expression (1.18), we obtain

$$\zeta_3 = -\frac{iQ\sigma^3}{8\pi^2 g^2} e^{i\sigma t} \left\{ \int_{-\infty}^{\infty} \left[\frac{\cos {}^1/_2 (\alpha - \theta) \cosh({}^1/_2 \eta)}{\cos^2 [{}^1/_2 (\alpha - \theta)] \cosh^2 ({}^1/_2 \eta) + \sin^2 [{}^1/_2 (\alpha - \theta)] \sinh^2 ({}^1/_2 \eta)} + \right.\right.$$
$$\left. + \frac{\cos {}^1/_2 (\alpha + \theta) \cosh({}^1/_2 \eta)}{\cos^2 [{}^1/_2 (\alpha + \theta)] \cosh^2 ({}^1/_2 \eta) + \sin^2 [{}^1/_2 (\alpha + \theta)] \sinh^2 ({}^1/_2 \eta)} \right] d\eta \times$$
$$\times \int_0^{\infty} \exp\left(-\frac{\sigma^2 h}{g} \xi\right) J_0\left(\frac{\sigma^2}{g} R_\eta \xi\right) \frac{\xi \, d\xi}{\xi - 1} \tag{2.6}$$

For the inner integral for large values of the quantity $\sigma^2 R_\eta/g$ (and, consequently, the more so for large values of $\sigma^2 r_0/g$) we have the asymptotic approximation

$$J_1 = \int_0^{\infty} \exp\left(-\frac{g^2 h}{g} \xi\right) J_0\left(\frac{\sigma^2}{g} R_\eta \xi\right) \frac{\xi \, d\xi}{\xi - 1} =$$

$$= -\pi i \exp\left(-\frac{\sigma^2 h}{g}\right) H_0^{(2)}\left(\frac{\sigma^2}{g} R_\eta\right) + O\left[\left(\frac{\sigma^2 r_0}{g}\right)^{-2}\right] \tag{2.7}$$

Taking account only of the first term of the asymptotic series, we obtain the following expression for ζ_3:

$$\zeta_3 = -\frac{Q\sigma^3}{8\pi g^2} e^{i\sigma t} \exp\left(-\frac{\sigma^2 h}{g}\right)\Big\{\cos\left[{}^1/_2(\alpha-\theta)\right]\times$$

$$\times \int\limits_{-\infty}^{\infty} \frac{\cosh({}^1/_2\eta) H_0^{(2)}(\sigma^2 R_\eta / g)\, d\eta}{\cos^2[{}^1/_2(\alpha-\theta)]\cosh^2({}^1/_2\eta) + \sin^2[{}^1/_2(\alpha-\theta)]\sinh^2({}^1/_2\eta)} +$$

$$+ \cos[{}^1/_2(\alpha+\theta)] \int\limits_{-\infty}^{\infty} \frac{\mathrm{ch}({}^1/_2\eta) H_0^{(2)}(\sigma^2 R_\eta / g)\, d\eta}{\cos^2[{}^1/_2(\alpha+\theta)]\cosh^2({}^1/_2\eta) + \sin^2[{}^1/_2(\alpha+\theta)]\quad{}^2({}^1/_2\eta)}\Big\} \tag{2.8}$$

Let us consider one of the integrals of Formula (2.8), for example, the first:

$$J_1 = \int\limits_{-\infty}^{\infty} \frac{\cosh({}^1/_2\eta) H_0^{(2)}(\sigma^2 R_\eta / g)}{\cos^2[{}^1/_2(\alpha-\theta)]\cosh^2({}^1/_2\eta) + \sin^2[{}^1/_2(\alpha-\theta)]\sinh({}^1/_2\eta)}\, d\eta \tag{2.9}$$

Changing the variable of integration according to the formula

$$\cosh\eta = 1 + \beta^2$$

we obtain

$$J_1 \frac{8}{\sqrt{2}} \int\limits_0^{\infty} \frac{H_0^{(2)}(\sigma^2 R_\eta / g)}{\cos^2[{}^1/_2(\alpha-\theta)](2+\beta^2) + \beta^2 \sin^2[{}^1/_2(\alpha-\theta)]}\, d\beta \tag{2.10}$$

Putting R_η in the form

$$R_\eta^2 = 2\rho\rho' r_0^2\left[\frac{(\rho+\rho')^2}{2\rho\rho'} + \beta^2\right] \tag{2.11}$$

and replacing the Haenkel function by the first term of its asymptotic representation

$$H_0^{(2)}(z) \approx \sqrt{\frac{2}{\pi z}}\, e^{-i(z - {}^1/_4\pi)}$$

the integral J_1 may be written in the form

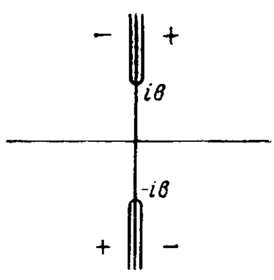

Fig. 3.

$$J_1 = \frac{8}{\pi} e^{i\frac{\pi}{4}} \left(\frac{\sigma^2 r_0}{g}\right)^{-1/2} (2\rho\rho')^{-1/4} \int\limits_0^{\infty} F(\beta)\, e^{-\omega w(\beta)}\, d\beta \tag{2.12}$$

where

$$F(\beta) = (b^2+\beta^2)^{-1/4} \frac{1}{\cos^2[{}^1/_2(\alpha-\theta)](\beta^2+2)+\beta^2\sin^2[{}^1/_2(\alpha-\theta)]}$$

$$w(\beta) = i\sqrt{2\rho\rho'}\sqrt{b^2+\beta^2}, \qquad b^2 = \frac{(\rho+\rho')^2}{2\rho\rho'}, \qquad \omega = \frac{\sigma^2 r_0}{g} \tag{2.13}$$

The asymptotic evaluation of the integral in Formula (2.12) may be obtained by the cross-over method. We will consider β a complex variable and determine the sign of the root $\chi = \sqrt{(b^2 + \beta^2)}$ when we introduce cuts from the branch points $\beta = \pm\, ib$ to infinity parallel to the imaginary axis. We assume, also, that on the first sheet, where the integration is carried out, the distribution of signs of the roots is as indicated in Fig. 3.

The cross-over point is located at the origin of coordinates, and the path (Fig. 4) along which the imaginary part of the function $w(\beta)$ remains constant as one moves away from the origin of coordinates, asymptotically approaches the straight line Re $\beta = b$. We deform the original path of integration *OA*, and replace it by the path *OBA*, where *BA* is a circular arc of infinitely large radius. Furthermore, expanding the integrand into a series near the cross-over point [4] and applying for its evaluation the lemma of Poincaré [4,5], we obtain as the result of all these considerations the asymptotic approximation of the integral J_1 in the form

$$J_1 = \frac{2}{\sqrt{\pi}} \frac{g}{\sigma^2 r_0 \sqrt{\rho\rho^1}} \exp\left[-i\,\frac{\sigma^2 r_0(\rho+\rho')}{g}\right] \frac{1}{\cos^2[{}^1/_2(\alpha-\theta)]} + O\left(\frac{\sigma^2 r_0}{g}\right)^{-2} \tag{2.14}$$

The evaluation of the second integral in Formula (2.8) is carried out in an analogous manner. As a result, for the term ζ_3 of the displacement of the liquid ζ we obtain

$$\zeta_3 = -\frac{Q\sigma}{2\pi\sqrt{\pi g}} \frac{1}{\sqrt{rr'}} \exp\left(-\frac{\sigma^2 h}{g}\right) \times$$
$$\times \exp\left\{i\sigma\left[t - \frac{\sigma(r+r')}{g}\right]\right\} \frac{\cos({}^1/_2\alpha)\cos({}^1/_2\theta)}{\cos[{}^1/_2(\alpha-\theta)]\cos[{}^1/_2(\alpha+\theta)]} \tag{2.15}$$

For α or θ equal to π, the elevation $\zeta_3 = 0$. This means (in the previously indicated approximation) that if the source or point of observation is in the half-plane relative to which diffraction occurs the term ζ_3 vanishes. The phase of the diffracted wave is determined by the total distance from the origin of the point of observation and of the source, and the damping in amplitude by the square root of the product of these distances. It should be noticed that for angles θ close to $\pi - \alpha$ and $\pi + \alpha$ the asymptotic expression (2.15) loses its meaning.

In conclusion, we note that a similar method may be used for the solution of the problem of waves initiated at the surface of a liquid by an oscillating body located near a wall immersed in the liquid. For this, one should use the method of N.E. Kochin, which replaces the oscillating body by a distribution of sources and sinks.

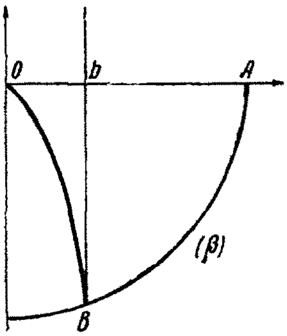

Fig. 4.

The author also solved the problem of waves on a surface of separation, when the source is located near a solid wall immersed in a half-space of fluid which consists of a homogeneous liquid, with another liquid of different density floating above it. Omitting here the rather cumbersome calculations, we present one of the formulas obtained. If the plane $z = 0$ is the undisturbed boundary of separation, if the layer has a thickness h, and the depth of the immersed source beneath the surface of separation is H, the portion of the displacement of the free surface ζ_{13} caused by diffraction is obtained in the following form:

$$\zeta_{13} = -\frac{Q\sigma}{2\pi\sqrt{\pi g}} \frac{\cos(^1/_2\,\alpha)\cos(^1/_2\,\theta)\,e^{i\sigma t}}{\cos[^1/_2(\alpha-\theta)]\cos[^1/_2(\alpha+\theta)]} \left\{\frac{\exp(-\sigma^2 H/g)\exp[-i(r+r')\sigma^2/g]}{\cosh(\sigma^2 h/g)-(1-2\alpha)\sinh(\sigma^2 h/g)} + \right.$$
$$\left. + \frac{g}{\sigma^2 h}\frac{\exp[-(\sigma^2 H/g)\xi_0]\exp[-i(\sigma^2/g)\xi_0(r+r')]}{(\xi_0-1)\{1-[(1-\alpha)\xi_0-\alpha]^2-(g/\sigma^2 h)(1-\alpha)\}\sinh[(\sigma^2 h/g)\xi_0]}\right\}\frac{1}{\sqrt{rr'}}$$

Here $\alpha = \rho_1/\rho$ is the ratio of the density of the layer to the density of the liquid within the half-space below, and ξ_0 is the root of the transcendental equation

$$\tanh\frac{\sigma^2 h}{g}\,\xi = \frac{1}{(1-\alpha)\,\xi-\alpha}$$

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